An Approach to a Generalized Rössler System via Mode Analysis*

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Z. Naturforsch. **50 a,** 1135–1138 (1995); received August 16, 1995

We investigate the generalized Rössler system introduced by Baier and Saale. They have linearly coupled additional degrees of freedom to the original Rössler system in order to construct a set of equations which shows maximum instability at an arbitrary dimension N, i. e., $N-2$ positive Lyapunov exponents. We present a transformation into a mode picture. It enables a qualitative understanding of the mechanism generating the dynamics of this complex system. The advantages of our approach are demonstrated for the exemplary case $N = 5$.

1. The generalized Rössler system

On their search for an N-dimensional model system that enables the investigation of the full chaotic hierarchy [1, 2] ranging from chaos with one positive Lyapunov exponent to hyperchaos with the maximum number of $N-2$ positive Lyapunov exponents, Baier and Sahle introduced the generalized Rössler system [3]. They have inserted into the original Rössler system [4] additional equations of the type $\dot{x}_n = x_{n-1} - x_{n+1}$ between the autocatalytic first equation and the nonlinear last equation, i. e., in matrix notation

$$\begin{aligned} \dot{\boldsymbol{x}} &= \mathbf{A}\boldsymbol{x} + \boldsymbol{e}_{N-1}\, x_N, \\ \dot{x}_N &= \epsilon + b x_N (x_{N-1} - d), \end{aligned} \qquad (1)$$

where
$\boldsymbol{x} = (x_1, x_2, \cdots, x_{N-1})^{\mathrm{T}}$, $\boldsymbol{e}_{N-1} = (0, \cdots, 0, 1)^{\mathrm{T}}$, and

$$\mathbf{A} = \begin{pmatrix} a & -1 & 0 & \dots & 0 \\ 1 & 0 & -1 & \ddots & \vdots \\ 0 & 1 & \ddots & -1 & 0 \\ \vdots & \ddots & 1 & 0 & -1 \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix}.$$

For $N = 3$, this is again the original Rössler system. The equations have the structure of a linear chain $\boldsymbol{x}$ with an autocatalytic term at one end, which is nonlinearly coupled to a trigger variable x_N at the other end (Fig. 1(a)).

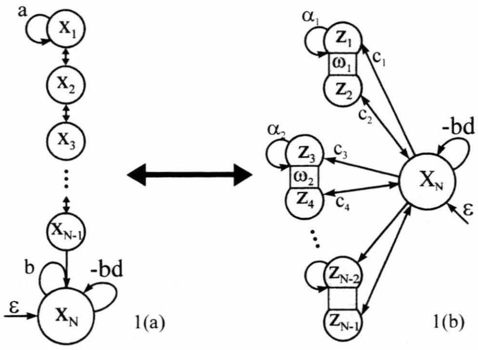

Fig. 1. The structure of the generalized Rössler system (a) in its original form (1), (b) in the mode picture ((2), (3)).

The system behaves qualitatively as follows: As long as x_{N-1} is smaller than the threshold d, x_N remains small (approximately $\epsilon/b(d - x_{N-1})$). Only if the system has gained enough energy (we call $\boldsymbol{x}^2$ energy) through the autocatalytic first variable x_1 such that x_N becomes larger than d, x_N starts to grow rapidly and folds the system back towards the center, thereby dissipating energy out of the system. Here we have the same stretching and folding mechanism as known from the original Rössler system. Thus one observes long intervals of smooth oscillation and growth of the linear subsystem, interrupted by short spikes of the trigger variable x_N which prevent the system from escaping to infinity.

2. The mode picture

In the following we restrict ourselves to the case of N being an odd number, where the linear chain has

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* Paper presented at the 5th Annual Meeting of ENDADYN, Grenoble, October 10 - 13, 1994.

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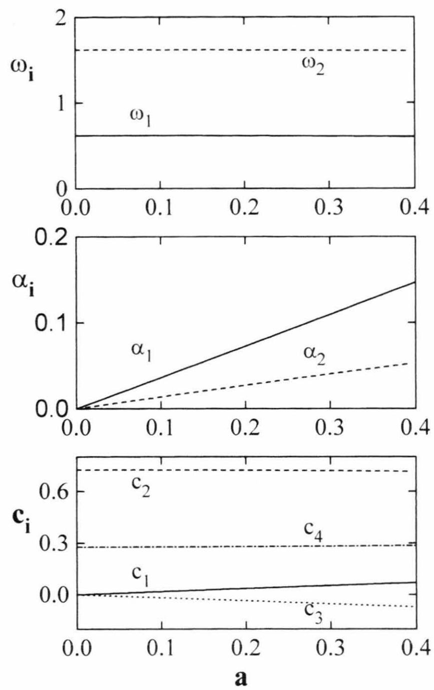

Fig. 2. The frequencies ω_i, the autocatalytic growth coefficients α_i, and the coupling constants c_i in the mode picture ((2), (3)) as a function of a ($\epsilon = 0.1, b = 4, d = 2$).

an even number of degrees of freedom and, therefore, $(N-1)/2$ pairs of complex conjugate eigenvalues. We present a real transformation $\boldsymbol{z} = \mathbf{U}^{-1}\boldsymbol{x}$ of the linear subsystem $\boldsymbol{x}$ into modes or oscillators of frequencies ω_i which are autocatalytic with growth coefficients α_i and interact only via the trigger variable x_N. The transformed equations read as follows:

$$\dot{\boldsymbol{z}} = \mathbf{B}\boldsymbol{z} + x_N \boldsymbol{c}, \quad (2)$$

$$\dot{x}_N = \epsilon - b x_N (z_2 + z_4 + \ldots + z_{N-1} - d), \quad (3)$$

where

$$\mathbf{B} = \mathbf{U}^{-1}\mathbf{A}\mathbf{U} = \begin{pmatrix} \alpha_1 & -\omega_1^2 & & & \\ 1 & 0 & & & \\ & & \alpha_2 & -\omega_2^2 & \\ & & 1 & 0 & \\ & & & & \ddots \end{pmatrix}$$

and $\boldsymbol{c} = \mathbf{U}^{-1} e_{N-1}$ are the coupling coefficients of x_N to $\boldsymbol{z}$. The frequencies ω_i, the growth coefficients α_i of the autocatalytic process, and the coupling coefficients c_i which govern the influence of x_N on the

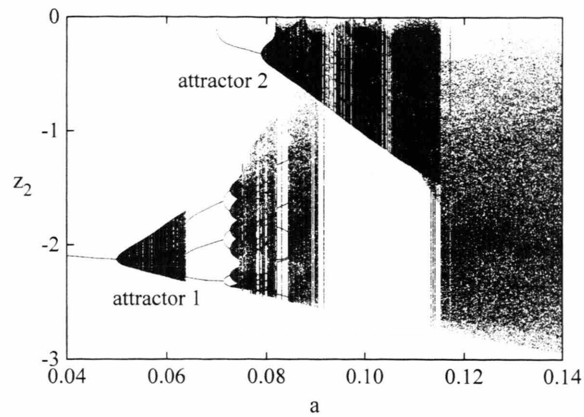

Fig. 3. Bifurcation diagram of the generalized Rössler system in the mode picture. The Poincaré section was calculated at $z_1 = 0$ (and $\dot{z}_1 < 0$). The diagram was computed with fixed initial values ($\boldsymbol{z}(0) = (1, 0, 0, 0, 0)$ and $\boldsymbol{z}(0) = (0, 0, 1, 0, 0)$) and a transient time of 10000 (corresponding to about 1000 revolutions).

oscillators are uniquely determined (when taking this special form of the nonlinear equation) and depend only on a. In the mode picture, all oscillators, (2), are coupled to the trigger variable x_N, (3), in the same manner. This symmetry is broken by the differences in the coefficients ω_i, α_i, and c_i belonging to the two oscillators. The underlying structure of the system in the mode picture is sketched in Figure 1(b).

3. Numerical results for $N = 5$

We have numerically investigated the system with $N = 5$ ($b = 4, d = 2, e = 0.1, a$ varying from 0 up to about 0.33). In the mode picture, this is equivalent to two autocatalytic oscillators (z_1, z_2) and (z_3, z_4) coupled via a nonlinear variable x_N. Oscillator (z_1, z_2) has a smaller frequency and a larger growth coefficient than oscillator (z_3, z_4). The dependence of the parameters ω_i, α_i, and c_i on a is depicted in Figure 2.

3.1. The two attractors

One could expect that oscillator (z_1, z_2) dominates the other one because it grows much faster and, thus, can trigger the spikes of x_N, i. e., the dissipation process, before (z_3, z_4) has a chance of developing in any way. In fact, this does actually happen for small values of a. We have observed a pronounced master-slave relationship between the oscillators (z_1, z_2) and (z_3, z_4). However, as can be seen in the bifurcation

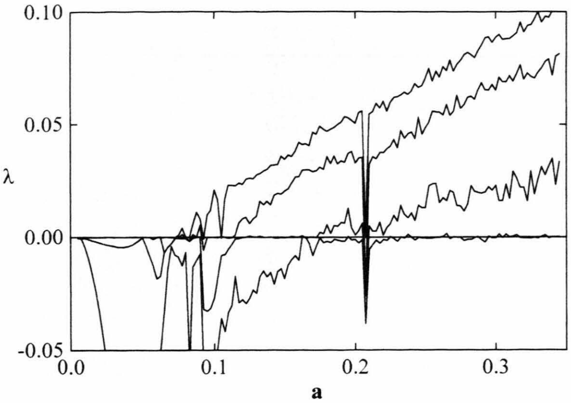

Fig. 4. Lyapunov spectrum of the generalized Rössler equations with $z(0) = (1, 0, 0, 0, 0)$, and a transient time of 20000. The fifth exponent remains below -6 and, thus, cannot be seen in this figure.

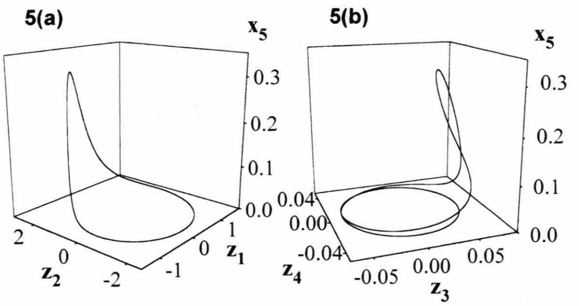

Fig. 5. Projections of the periodic orbit for a = 0.04 (ϵ = 0.1, b = 4, d = 2) on attractor 1 onto (a) (z_1, z_2) (master) and x_N, (b) (z_3, z_4) (slave) and x_N.

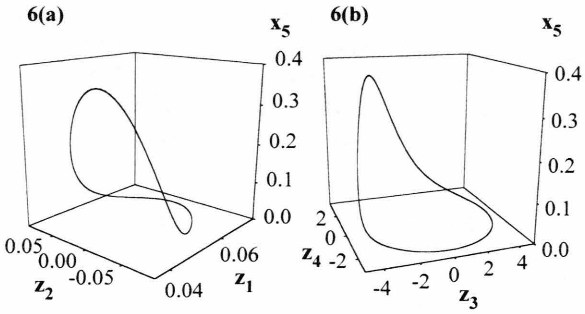

Fig. 6. Projections of the periodic orbit for a = 0.04 (ϵ = 0.1, b = 4, d = 2) on attractor 2 onto (a) (z_1, z_2) (slave) and x_N, (b) (z_3, z_4) (master) and x_N.

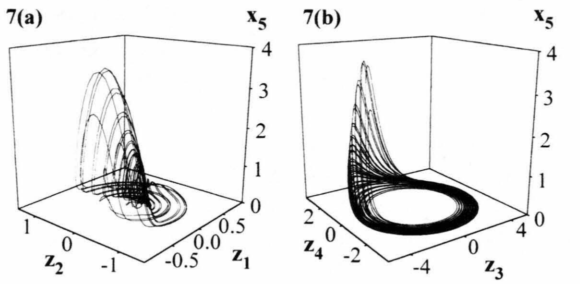

Fig. 7. Projections of the chaotic orbit with one positive Lyuapunov exponent on attractor 2 for a = 0.10 (ϵ = 0.1, b = 4, d = 2) onto (a) (z_1, z_2) (slave) and x_N, (b) (z_3, z_4) (master) and x_N.

diagram (Fig. 3), there is another coexisting attractor which corresponds to oscillator (z_3, z_4) being the master and oscillator (z_1, z_2) being the slave. In the following the two attractors will be called attractor 1 and 2, as it is the case in Figure 3.

The existence of the two attractors becomes evident if one considers the perfectly symmetric version of the system ($\omega_1 = \omega_2$, $\alpha_1 = \alpha_2$, $c_1 = c_3$, $c_2 = c_4$). In this case, each solution has a complement solution which is obtained by the transformation $z_1 \rightarrow z_3$, $z_2 \rightarrow z_4$, $z_3 \rightarrow z_1$, $z_4 \rightarrow z_2$. Consequently, for each attractor there must be a complement attractor obtained by the same transformation. In our asymmetric oscillator the fundamental structure of the symmetric case seems to be preserved as there are two attractors. There is always one dominant oscillator with higher amplitude that triggers the variable x_N and whose phase is only marginally influenced by the spikes of x_N.

3.2. On a journey to higher a: from periodicity to hyperchaos

In the following we discuss the development of attractor 1 (i. e., the one where oscillator (z_1, z_2) is dominant), under variation of a. For $a \in [0, 0.006]$ there is a stable fixed point. At a = 0.006 (see the Lyapunov spectrum in Fig. 4) it becomes unstable and a stable limit cycle emerges. As one can see in Figs. 5(a) and 5(b), oscillator (z_1, z_2) is dominant. It has a much larger amplitude than the other one and it triggers the spikes of the variable x_N, which in turn shift the phase of the other one in such a way that a 1:2 locking arises. One may say that oscillator (z_3, z_4) is slaved, since it virtually does not influence the motion of oscillator (z_1, z_2). For a = 0.051, the system can no longer sustain the mode locking, and a quasiperiodic state appears. As a grows further, we get a window with a 3:8 mode locking which bifurcates to a 6:16 mode locking through period doubling. Eventually we enter the chaotic regime via a complex interwoven structure of highly periodic, quasiperiodic, and chaotic intervals.

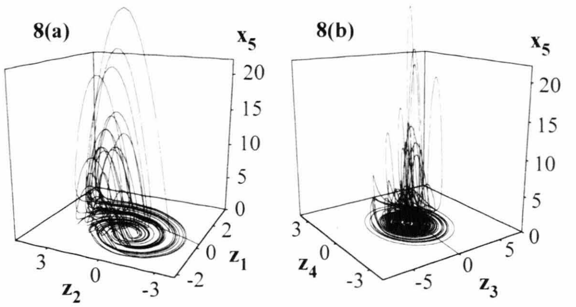

Fig. 8. Projections of the chaotic orbit with three positive Lyapunov exponents for $a = 0.30$ ($\epsilon = 0.1, b = 4, d = 2$) onto (a) (z_1, z_2) (slave) and x_N, (b) (z_3, z_4) (master) and x_N.

Up to $a = 0.09$, attractor 2 (the one where oscillator (z_3, z_4) is dominant) develops in a largely analogous way, exept with exchanged roles of the two oscillators, i. e., here oscillator (z_3, z_4) is dominant in the above sense and the other one is slaved (see Figs. 4, 6, and 7). In the interval $a \in [0.09, 0.115]$ we have found only attractor 2, and for $a > 0.115$ only one attractor. This phenomenon may be discussed in terms of two crises. It is interesting to note that the second largest Lyapunov exponent becomes positive just at $a = 0.115$.

For $a > 0.115$ we already have two positive Lyapunov exponents. In that region the three largest exponents seem to grow almost linearly with a, which leads to the third positive exponent at approximately $a = 0.17$ (see Figure 8). We could not see any qualitative change of the behavior of the system at the point where the third exponent becomes positive, neither in the bifurcation diagram nor in the trajectories. The motion of the system rather seems to gain complexity continuously as one goes to higher a. Especially the master-slave relationship levels out more and more.

4. Conclusion and outlook

We have introduced a transformation of the generalized Rössler system into a mode picture, where we have autocatalytic linear oscillators coupled only via a nonlinear trigger variable. This picture possesses an inherent symmetry due to the fundamental equivalence of the oscillators. This symmetry is revealed in the coexistence of two attractors in the case of $N = 5$, where we have two oscillators. The dynamics on the attractors is governed by a master-slave relationship between the two oscillators. As one goes from one attractor to the other, the two oscillators exchange their roles. In our case, the symmetry is broken because the oscillators have different parameters like frequencies etc., which complicates matters considerably. Therefore it should be instructive to investigate a system with identical oscillators having the full symmetry. In higher dimension, our approach may prove to be particularly powerful, as in a system of n oscillators there are $n!$ symmetries corresponding to $n!$ different transpositions of two oscillators. To conclude, we state that all linear systems with one localized nonlinearity, including discretized delay equations, can be transformed into a mode picture and can, therefore, be analyzed accordingly.

Acknowledgements

We would like to thank G. Baier and S. Sahle for fruitful discussions and the Deutsche Forschungsgemeinschaft for financial support.

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